

Maximal Subsemigroups containing a particular semigroup

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Abstract

We study maximal subsemigroups of the monoid $T(X)$ of all full transformations on the set $X = N$ of natural numbers containing a given subsemigroup W of $T(X)$ where each element of a given set U is a generator of $T(X)$ modulo W . This note continues the study of maximal subsemigroups on the monoid of all full transformations on an infinite set.

In this note, we want to continue the study of maximal subsemigroups of the semigroup $T(X)$ of all full transformations on an infinite set, in particular, for the case X is countable. The maximal subsemigroups of $T(X)$ containing the symmetric group $Sym(X)$ of all bijective mappings on an infinite set are already known. They were determined by L. Heindorf (X is countable) and by M. Pinsker (any infinite set X) characterizing maximal clones ([3],[6]).

The setwise stabilizer of any finite set $Y \subseteq X$ under $Sym(X)$ is a subgroup of $Sym(X)$. In [1], the authors determine the maximal subsemigroups of $T(X)$ containing the setwise stabilizer of any finite set $Y \subseteq X$ under $Sym(X)$. For a finite partition of X , one can also consider the (almost) stabilizer. They form subsemigroups of $Sym(X)$ and in [1], the maximal subsemigroups of $T(X)$ containing such a subgroup are determined. Also in [1], the maximal subsemigroups containing the stabilizer of any uniform ultrafilter on X , which forms also a group, are determined.

In the present note, we consider a countable infinite set X and characterize for a given subsemigroup W of $T(X)$ and a given set $U \subseteq T(X)$, where any element of U is a generator modulo W (see [4]), the maximal subsemigroups of $T(X)$ containing W . As a consequence of this result, we obtain all maximal subsemigroups of $T(X)$ containing $T(X) \setminus S$, where S is a given maximal subsemigroup of $T(X)$ containing $Sym(X)$.

Notation 1. Let $M \subseteq P(T(X))$ and let $J(M)$ be the set of all $A \subseteq T(X)$ with
 $A \subseteq \bigcup M := \{a \mid \exists m \in M \wedge a \in m\},$
 $A \cap m \neq \emptyset$ for all $m \in M$, and
 $\forall \alpha \in A \exists m \in M$ with $A \cap m = \{\alpha\}.$

Definition 2. Let $U \subseteq T(X)$ and $W \leq T(X)$. Then we put
 $Gen(U) := \{A \subseteq T(X) \mid A \text{ is finite and } \langle A \rangle \cap U \neq \emptyset\}$ and
 $\mathcal{H}(U, W) := \{A \subseteq T(X) \setminus W \mid A \in J(Gen(U))\}.$

Theorem 3. Let $W \leq S \leq T(X)$ and $U \subseteq T(X)$ with $U \cap W = \emptyset$ such that $\langle W, \alpha \rangle = T(X)$ for all $\alpha \in U$. Then the following statements are equivalent:

(i) S is maximal.

(ii) There is a $H \in \mathcal{H}(U, W)$ with $S = T(X) \setminus H$.

Proof. (i) \Rightarrow (ii): Assume that $S \cap H \neq \emptyset$ for all $H \in \mathcal{H}(U, W)$. Then there is $A \in \text{Gen}(U)$ with $A \subseteq S$. Otherwise $A \not\subseteq S$, i.e. $A \setminus S \neq \emptyset$, for all $A \in \text{Gen}(U)$. We consider the set $\text{Gen}(U) = \{A_i | i \in \mathbb{N}\}$. We put $\bar{A}_i = A_i \setminus S$ for all $i \in \mathbb{N}$,

$$H_0 = \emptyset \text{ and } \bar{H}_0 = \emptyset;$$

$$H_{i+1} = H_i \text{ if } \bar{A}_{i+1} \cap H_i \neq \emptyset \text{ or } \exists j > i + 1 \text{ with } \bar{A}_j \setminus \bar{H}_i \subseteq \bar{A}_{i+1} \text{ then we define}$$

$$\bar{H}_{i+1} = \bar{H}_i;$$

$$H_{i+1} = H_i \cup \{a_{i+1}\} \text{ if } \bar{A}_{i+1} \cap H_i = \emptyset \text{ and } \forall j > i + 1, \bar{A}_j \setminus \bar{H}_i \not\subseteq \bar{A}_{i+1} \text{ then we define}$$

$$\bar{H}_{i+1} = (\bar{H}_i \cup \bar{A}_{i+1}) \setminus \{a_{i+1}\} \text{ for } a_{i+1} \in \bar{A}_{i+1} \setminus \bar{H}_i.$$

We want to show that $\bar{A}_{i+1} \setminus \bar{H}_i \neq \emptyset$ for all $i \in \mathbb{N}$. Let us consider $i \in \mathbb{N}$. We know that $\bar{A}_i = A_i \setminus S \neq \emptyset$ by assumption. If $H_i = \emptyset$ then $\bar{A}_{i+1} \setminus \bar{H}_i \neq \emptyset$. Assume that $H_i \neq \emptyset$. Then there exists $k \in \mathbb{N}$ with $k < i$ such that $H_k \cup \{a_{k+1}\} = H_{k+1} = H_i$. If $\bar{A}_{i+1} \setminus \bar{H}_k \subseteq \bar{A}_{k+1}$ then $\bar{H}_{k+1} = \bar{H}_k = \bar{H}_i$. This implies that $\bar{A}_{i+1} \setminus \bar{H}_i \neq \emptyset$. Then $\bar{A}_j \setminus \bar{H}_k \not\subseteq \bar{A}_{k+1}$ for all $j > k + 1$. Thus $\bar{A}_{i+1} \setminus \bar{H}_k \not\subseteq \bar{A}_{k+1}$ and there exists $x \in \bar{A}_{i+1}$ but $x \notin \bar{H}_k$ and $x \notin \bar{A}_{k+1}$, i.e. $x \in \bar{A}_{i+1}$ but $x \notin \bar{H}_k \cup \bar{A}_{k+1}$. Moreover, we have $x \in \bar{A}_{i+1}$ but $x \notin (\bar{H}_k \cup \bar{A}_{k+1}) \setminus \{a_{k+1}\} = \bar{H}_i$. This shows that $x \in \bar{A}_{i+1} \setminus \bar{H}_i$. This completes the proof that $\bar{A}_{i+1} \setminus \bar{H}_i \neq \emptyset$. Moreover, we put $H := \bigcup_{i \in \mathbb{N}} H_i$.

We want to show that $H := \bigcup_{i \in \mathbb{N}} H_i \in J(\text{Gen}(U))$. First, we will show that $H \subseteq \bigcup \text{Gen}(U) := \{a_i | \exists A_i \in \text{Gen}(U) \wedge a_i \in A_i\}$. By definition of H , we know that $H_0 = \emptyset$, $H_{i+1} = H_i$ or $H_{i+1} = H_i \cup \{a_{i+1}\}$ for $i \in \mathbb{N}$. This shows that $H \subseteq \bigcup \text{Gen}(U)$.

Let $i \in \mathbb{N}$. Then we have to show that $H \cap A_i \neq \emptyset$. If $a_i \in H$ then all is clear. Otherwise, assume that $a_i \notin H$. Then, $\bar{A}_i \cap H_{i-1} \neq \emptyset$ and thus $\bar{A}_i \cap H_i \neq \emptyset$ or $\exists k > i$ with $\bar{A}_k \setminus \bar{H}_{i-1} \subseteq \bar{A}_i$. We have to consider \bar{A}_k with $k > i$. Then $a_k \in H$ or we have the same cases as for A_i . Since the elements of $\text{Gen}(U)$ are finite, this procedure finish. So we obtain an $s \geq i$ such that $a_s \in H$ and it is routine to see that also $a_s \in \bar{A}_i$. Thus $H \cap A_i \neq \emptyset$.

Let $a \in H$. Then there exists an $i \in \mathbb{N}$ such that $a = a_i \in A_i$. We have to show that for all $k \in \mathbb{N} \setminus \{i\}$, if $a_k \in H$ (i.e. $a_k \in A_k$) then a_k does not belong to A_i . We have two cases:

Case 1: $k > i$. Then a_k does not belong to $H_i \subseteq H_{k-1}$ and moreover, it does not belong to $\bar{H}_i \subseteq \bar{H}_{k-1}$. But $\bar{A}_i \subseteq H_i \cup \bar{H}_i$. So a_k does not belong to A_i .

Case 2: $k < i$. Then $a_k \in H_{i-1}$ where $H_{i-1} \cap A_i = \emptyset$, whence $a_k \notin A_i$.

This shows that $H \in J(\text{Gen}(U))$. But $H \cap S = \emptyset$ is a contradiction. Then there is $A \in \text{Gen}(U)$ with $A \subseteq S$, i.e. $\langle A \rangle \cap U \neq \emptyset$. Let $\alpha \in \langle A \rangle \cap U \subseteq S$. Then $T(X) = \langle W, \alpha \rangle \leq S$, i.e. $S = T(X)$, a contradiction. Hence, there is $H \in \mathcal{H}(U, W)$ with $S \cap H = \emptyset$, i.e. $S \subseteq T(X) \setminus H$. We want to show that $T(X) \setminus H$ is a semigroup. Let $\alpha, \beta \in T(X) \setminus H$. Assume that $\alpha\beta \in H$. Then there is $A \in \text{Gen}(U)$ with $A \cap H = \{\alpha\beta\}$. Since $U \cap \langle A \rangle \neq \emptyset$, $U \cap \langle A \setminus \{\alpha\beta\} \cup \{\alpha, \beta\} \rangle \neq \emptyset$, i.e. $(A \setminus \{\alpha\beta\}) \cup \{\alpha, \beta\} \in \text{Gen}(U)$. Since $A \cap H = \{\alpha\beta\}$, $(A \setminus \{\alpha\beta\}) \cup \{\alpha, \beta\} \in \text{Gen}(U)$, implies $\alpha \in H$ or $\beta \in H$, a contradiction. Hence $T(X) \setminus H$

is a semigroup. Since S is a maximal subsemigroup of $T(X)$ and $S \subseteq T(X) \setminus H$ this implies $S = T(X) \setminus H$.

(ii) \Rightarrow (i): Let $H \in \mathcal{H}(U, W)$ with $S = T(X) \setminus H$. We have shown that $T(X) \setminus H$ is a semigroup. Now, we want to show that it is a maximal subsemigroup of $T(X)$. Let $\alpha \in H$. Then there is $A \in \text{Gen}(U)$ with $A \cap H = \{\alpha\}$ and $T(X) = \langle W, A \rangle \subseteq \langle T(X) \setminus H, A \rangle = \langle T(X) \setminus H, \alpha \rangle$ since $\langle A \rangle \cap U \neq \emptyset$ and $\langle W, \beta \rangle = T(X)$ for all $\beta \in U$. So, $\langle T(X) \setminus H, \alpha \rangle = T(X)$. This shows that $T(X) \setminus H$ is a maximal subsemigroup of $T(X)$. \square

If $\alpha \in T(X)$ and $A \subseteq X$ such that the restriction of α to A is injective and have the same range as α , then we will refer A as transversal of α ($\ker \alpha$ denotes the kernel of α). We will also write $A \# \ker \alpha$ if A is a transversal of α .

Let $D(\alpha) := X \setminus \text{im} \alpha$ ($\text{im} \alpha$ denotes the range of α). The rank α , i.e. the cardinality of $\text{im} \alpha$, is denoted by $\text{rank}(\alpha) := |\text{im} \alpha|$. Then $d(\alpha) := |D(\alpha)|$ is called defect of α and $c(\alpha) := \sum_{y \in \text{im} \alpha} (|y\alpha^{-1}| - 1)$ is called collapse of α .

Moreover, we put $K(\alpha) := \{x \in \text{im} \alpha \mid |x\alpha^{-1}| = \aleph_0\}$ and $k(\alpha) := |K(\alpha)|$ is called infinite contractive index. It is well known that $d(\alpha\beta) \leq d(\alpha) + d(\beta)$, $k(\alpha\beta) \leq k(\alpha) + k(\beta)$ [2] and $c(\alpha\beta) \leq c(\alpha) + c(\beta)$ [?] for $\alpha, \beta \in T(X)$. For more background in the theory of transformation semigroups see [2] and [5].

Now we want to determine the maximal subsemigroups of $T(X)$ containing $T(X) \setminus S$, where S is one of the five maximal subsemigroups of $T(X)$ containing $\text{Sym}(X)$. Let us introduce the following five sets:

- $\text{Inj}(X) := \{\alpha \in T(X) \mid \text{rank}(\alpha) = \aleph_0, c(\alpha) = 0 \text{ and } d(\alpha) \neq 0\}$ (the set of injective but not surjective mappings on X).
- $\text{Sur}(X) := \{\alpha \in T(X) \mid \text{rank}(\alpha) = \aleph_0, c(\alpha) \neq 0 \text{ and } d(\alpha) = 0\}$ (the set of surjective but not injective mappings on X).
- $C_p(X) := \{\alpha \in T(X) \mid \text{rank}(\alpha) = \aleph_0, k(\alpha) = \aleph_0\}$.
- $IF(X) := \{\alpha \in T(X) \mid \text{rank}(\alpha) = \aleph_0, c(\alpha) = \aleph_0 \text{ and } d(\alpha) < \aleph_0\}$.
- $FI(X) := \{\alpha \in T(X) \mid \text{rank}(\alpha) = \aleph_0, d(\alpha) = \aleph_0 \text{ and } c(\alpha) < \aleph_0\}$.

In [3], the following proposition was proved. Note that we independently proved this proposition whilst of the work of L. Heindorf. We thank Martin Goldstern for bringing these reference to our consideration at the AAA82 in Potsdam (June 2011). For the sake of completeness, we include the proof of this proposition.

Proposition 4. *The following semigroups of $T(X)$ are maximal:*

$$T(X) \setminus H$$

for $H \in \{\text{Inj}(X), \text{Sur}(X), C_p(X), IF(X), FI(X)\}$.

Proof. 1) Let $\alpha, \beta \in T(X) \setminus \text{Inj}(X)$. Assume that $\alpha\beta \in \text{Inj}(X)$. Then $c(\alpha) = 0$, i.e. α is injective. Since $\alpha \notin \text{Inj}(X)$, $\alpha \in \text{Sym}(X)$. But $c(\alpha\beta) = 0$ and $\alpha \in \text{Sym}(X)$ implies β is injective. Since $\beta \notin \text{Inj}(X)$, $\beta \in \text{Sym}(X)$. So $\alpha\beta \in \text{Sym}(X)$, i.e. $\alpha\beta$ is surjective, contradicts $\alpha\beta \in \text{Inj}(X)$. This shows that $T(X) \setminus \text{Inj}(X)$ is a semigroup.

Let $\alpha \in \text{Inj}(X)$. Then we will show that $\langle T(X) \setminus \text{Inj}(X), \alpha \rangle = T(X)$. For this let $\beta \in \text{Inj}(X)$. Let $a \in \text{im}\beta$. Let $\gamma \in T(X)$ with $x\gamma = a$ for $x \in D(\alpha)$, and $x\gamma = x\alpha^{-1}\beta$ for $x \in \text{im}\alpha$. Then $x\alpha\gamma = x\alpha\alpha^{-1}\beta = x\beta$ for all $x \in X$. This shows $\alpha\gamma = \beta$, where $\gamma \notin \text{Inj}(X)$ since $D(\alpha) \neq \emptyset$. This shows that $T(X) \setminus \text{Inj}(X)$ is maximal.

2) Let $\alpha, \beta \in T(X) \setminus \text{Sur}(X)$. Assume that $\alpha\beta \in \text{Sur}(X)$. Then $d(\beta) = 0$, i.e. β is surjective. Since $\beta \notin \text{Sur}(X)$, $\beta \in \text{Sym}(X)$. But $d(\alpha\beta) = 0$ and $\beta \in \text{Sym}(X)$ implies α is surjective. Since $\alpha \notin \text{Sur}(X)$, $\alpha \in \text{Sym}(X)$. So $\alpha\beta \in \text{Sym}(X)$, i.e. $\alpha\beta$ is injective, contradicts $\alpha\beta \in \text{Sur}(X)$. This shows that $T(X) \setminus \text{Sur}(X)$ is a semigroup.

Let $\alpha \in \text{Sur}(X)$. Then we will show that $\langle T(X) \setminus \text{Sur}(X), \alpha \rangle = T(X)$. For this let $\beta \in \text{Sur}(X)$. For all $\bar{x} \in X/\ker\alpha$ we fix a $\bar{x}^* \in \bar{x}$. Then we consider the following $\delta \in T(X)$ with $i\delta = (i\beta\alpha^{-1})^*$ for all $i \in X$. Hence $i\delta\alpha = i\beta$ for all $i \in X$. This shows $\delta\alpha = \beta$. Since $\text{im}\delta = \{\bar{x}^* \mid \bar{x} \in X/\ker\alpha\} \neq X$ (because α is not injective), $\delta \in T(X) \setminus \text{Sur}(X)$. This shows $\beta = \delta\alpha \in \langle T(X) \setminus \text{Sur}(X), \alpha \rangle$. Consequently, $\langle T(X) \setminus \text{Sur}(X), \alpha \rangle = T(X)$. This shows that $T(X) \setminus \text{Sur}(X)$ is maximal.

3) Let $\alpha, \beta \in T(X) \setminus C_p(X)$. Further, let $x \in \text{im}\alpha\beta$. Then $x(\alpha\beta)^{-1} = (x\beta^{-1})\alpha^{-1}$ and $|(x\beta^{-1})\alpha^{-1}| = \aleph_0$ if $|x\beta^{-1}| = \aleph_0$ or there is a $y \in x\beta^{-1} \cap \text{im}\alpha$ with $|y\alpha^{-1}| = \aleph_0$. This shows $k(\alpha\beta) \leq k(\alpha) + k(\beta) < \aleph_0 + \aleph_0 = \aleph_0$. This shows that $\alpha\beta \in T(X) \setminus C_p(X)$.

Let $\alpha \in C_p(X)$. Then, we will show that $\langle T(X) \setminus C_p(X), \alpha \rangle = T(X)$. For this let $\beta \in C_p(X)$. Then, there is a bijection

$$f : X/\ker\beta \rightarrow \{x\alpha^{-1} \mid x \in K(\alpha)\}.$$

For each $\bar{x} \in X/\ker\beta$, there is an injective mapping

$$f_{\bar{x}} : \bar{x} \rightarrow f(\bar{x}).$$

We take the $\gamma \in \text{Inj}(X)$ with $i\gamma = f_{\bar{x}}(i)$ where $i \in \bar{x}$ for $\bar{x} \in X/\ker\beta$. For $i, j \in X$, $i\beta = j\beta$ if and only if there is an $\bar{x} \in X/\ker\beta$ with $i, j \in \bar{x}$, i.e. $f_{\bar{x}}(i)\alpha = f_{\bar{x}}(j)\alpha$. But $f_{\bar{x}}(i)\alpha = f_{\bar{x}}(j)\alpha$ is equivalent to $i\gamma\alpha = j\gamma\alpha$, consequently, we have $i\gamma\alpha = j\gamma\alpha$ if and only if $i\beta = j\beta$. Further, let $\delta \in T(X)$ with $i\gamma\alpha\delta = i\beta$ for $i \in X$ and $i\delta = x_0$ (x_0 is any fixed element in X) for $i \in X \setminus \text{im}\gamma\alpha$. Since $i\gamma\alpha = j\gamma\alpha$ if and only if $i\beta = j\beta$, δ is well defined. Moreover, $|\{\bar{x} \mid \bar{x} \in X/\ker\delta, |\bar{x}| = \aleph_0\}| < 2$, i.e. $\delta \in T(X) \setminus C_p(X)$. This shows $\beta = \gamma\alpha\delta \in \langle T(X) \setminus C_p(X), \alpha \rangle$. Consequently, $\langle T(X) \setminus C_p(X), \alpha \rangle = T(X)$. This shows that $T(X) \setminus C_p(X)$ is maximal.

4) Let $\alpha, \beta \in T(X) \setminus IF(X)$.

If $c(\alpha) < \aleph_0$ and $c(\beta) < \aleph_0$ then $c(\alpha\beta) \leq c(\alpha) + c(\beta) < \aleph_0$, i.e. $\alpha\beta \notin IF(X)$.

If $d(\alpha) = \aleph_0$ and $c(\beta) < \aleph_0$ then $|\{\bar{x} \in X/\ker\beta \mid \bar{x} \cap \text{im}\alpha = \emptyset\}| = \aleph_0$. This implies $d(\alpha\beta) = \aleph_0$, i.e. $\alpha\beta \notin IF(X)$.

If $d(\beta) = \aleph_0$ then $d(\alpha\beta) \geq d(\beta) = \aleph_0$, i.e. $\alpha\beta \notin IF(X)$.

Altogether, this shows that $\alpha\beta \in T(X) \setminus IF(X)$.

Let $\alpha \in IF(X)$. Then we will show that $\langle T(X) \setminus IF(X), \alpha \rangle = T(X)$. For this let $\beta \in IF(X)$. Let $\gamma \in T(X)$ with $\ker\gamma = \ker\beta$ and $\text{im}\gamma \# \ker\alpha$. For each $\bar{x} \in X/\ker\beta$,

we fix any \bar{x}^* . Since $c(\alpha) = \aleph_0$, $d(\gamma) = \aleph_0$, i.e. $\gamma \notin IF(X)$. Further, let $\delta \in T(X)$ with $im\alpha \# \ker \delta$ and $i\delta = (i(\gamma\alpha)^{-1})^*\beta$ for $i \in im\alpha$. Since $im\gamma \# \ker \alpha$, we have $im\alpha = im\gamma\alpha$ and δ is well defined. Because of $im\gamma \# \ker \alpha$, $\ker \gamma\alpha = \ker \gamma = \ker \beta$, where $i\gamma\alpha\delta = (i\gamma\alpha(\gamma\alpha)^{-1})^*\beta = i\beta$ for $i \in X$. Note that from $im\alpha \# \ker \delta$ and $d(\alpha) < \aleph_0$, it follows $c(\delta) < \aleph_0$, i.e. $\delta \notin IF(X)$. This shows $\beta = \gamma\alpha\delta \in \langle T(X) \setminus IF(X), \alpha \rangle$. Consequently, $\langle T(X) \setminus IF(X), \alpha \rangle = T(X)$. This shows that $T(X) \setminus IF(X)$ is maximal.

5) Let $\alpha, \beta \in T(X) \setminus FI(X)$.

If $c(\alpha) = \aleph_0$ then $c(\alpha\beta) \geq c(\alpha) = \aleph_0$, i.e. $\alpha\beta \notin FI(X)$.

If $d(\alpha) < \aleph_0$ and $c(\beta) = \aleph_0$ then $|\{i \in im\alpha \mid \exists j \in im\alpha \setminus \{i\} \text{ such that } i\beta = j\beta\}| = \aleph_0$. This implies $c(\alpha\beta) = \aleph_0$, i.e. $\alpha\beta \notin FI(X)$.

If $d(\alpha) < \aleph_0$ and $d(\beta) < \aleph_0$ then $d(\alpha\beta) \leq d(\alpha) + d(\beta) < \aleph_0$, i.e. $\alpha\beta \notin FI(X)$.

Altogether, this shows that $\alpha\beta \in T(X) \setminus FI(X)$.

Let $\alpha \in FI(X)$. Then, we will show that $\langle T(X) \setminus FI(X), \alpha \rangle = T(X)$. For this, let $\beta \in FI(X)$. Let $\gamma \in T(X)$ with $\ker \gamma = \ker \beta$ and $im\gamma \# \ker \alpha$. For each $\bar{x} \in X/\ker \beta$, we fix any \bar{x}^* . Since $c(\alpha) < \aleph_0$, $d(\gamma) < \aleph_0$, i.e. $\gamma \notin FI(X)$. Further, let $\delta \in T(X)$ with $im\alpha \# \ker \delta$ and $i\delta = (i(\gamma\alpha)^{-1})^*\beta$ for $i \in im\alpha$. Since $im\gamma \# \ker \alpha$, $im\alpha = im\gamma\alpha$ and δ is well defined, where $i\gamma\alpha\delta = (i\gamma\alpha(\gamma\alpha)^{-1})^*\beta = i\beta$ for $i \in X$. Note that from $im\alpha \# \ker \delta$ and $c(\alpha) = \aleph_0$, it follows $d(\delta) = \aleph_0$, i.e. $\delta \notin FI(X)$. This shows $\beta = \gamma\alpha\delta \in \langle T(X) \setminus FI(X), \alpha \rangle$. Consequently, $\langle T(X) \setminus FI(X), \alpha \rangle = T(X)$. This shows that $T(X) \setminus FI(X)$ is maximal. \square

First, we characterize the maximal subsemigroups of $T(X)$ containing $Inj(X)$ and $Sur(X)$, respectively. Note that we do not need Theorem 3 here. It is well known that the set $F(X)$ of all transformation of a finite rank forms an ideal of $T(X)$ and $Inf(X) := T(X) \setminus F(X)$ generates $T(X)$. The next lemma shows that any maximal subsemigroup S of $T(X)$ has the form $S = F(X) \cup T$ for some $T \subset Inf(X)$.

Lemma 5. *Let S be a maximal subsemigroup of $T(X)$. Then $F(X) \subset S$.*

Proof. We have $Inf(X) \not\subseteq S$ (since $S \neq T(X)$). Since $F(X)$ forms an ideal of $T(X)$ and thus $Inf(X) \not\subseteq S$ implies $S \subseteq S \cup F(X) \neq T(X)$. Because of the maximality of S , we have $S = S \cup F(X)$, i.e. $F(X) \subset S$. \square

Lemma 6. *Let $Sur(X) \subset S \leq T(X)$ with $Inj(X) \cap S \neq \emptyset$ and $FI(X) \cap S \neq \emptyset$. Then*

$$S = T(X).$$

Proof. We have $F(X) \subset S$ by Lemma 5. Hence, we have to consider only the elements of $Inf(X)$. Let $\alpha \in Sym(X)$. Then there is a $\beta \in Inj(X) \cap S$ and we take the $\gamma \in Sur(X)$ with $i\gamma = i$ for $i \in D(\beta)$ and $i\beta\gamma = i\alpha$ for $i \in X$. Since $im\beta = X \setminus D(\beta)$, this shows that $\beta\gamma = \alpha$, and consequently, $Sym(X) \subset S$. Let us put

$$\begin{aligned} A & : = \{\alpha \in Inf(X) \mid d(\alpha) < \aleph_0\} \\ B & : = \{\alpha \in Inf(X) \mid d(\alpha) = \aleph_0\}. \end{aligned}$$

Clearly, $\text{Inf}(X) = A \cup B$. Let $\alpha \in A$. If $d(\beta) < \aleph_0$ then for each natural number $k \geq 1$, there is a natural number $r \geq 1$ such that $d(\beta^r) \geq k$. Since $\beta^r \in \text{Inj}(X) \cap S$, we can assume that $d(\beta) \geq d(\alpha)$. Since $d(\beta) \geq d(\alpha)$, there is a $\gamma_1 \in \text{Sur}(X)$ such that γ_1 restricted to $\text{im}\beta$ is bijective with $\text{im}\alpha$ as range and $D(\beta)\gamma_1 = D(\alpha)$. We take the $\gamma_2 \in \text{Inf}(X)$ with $i\gamma_2$ is the unique element in $i\alpha\gamma_1^{-1}\beta^{-1}$ for $i \in X$. Since β is injective, we have $\gamma_2 \in \text{Sur}(X) \cup \text{Sym}(X)$. Then we have $i\gamma_2\beta\gamma_1 = i\alpha\gamma_1^{-1}\beta^{-1}\beta\gamma_1 = i\alpha$ for $i \in X$. This shows $\alpha = \gamma_2\beta\gamma_1 \in S$, and consequently, $A \subset S$.

Let $\alpha \in B$. Moreover, there is a $\delta \in \text{FI}(X) \cap S$. Then there is a $\eta \in A$ with $\ker \alpha = \ker \eta$ and $\text{im}\eta \# \ker \delta$. Since $d(\alpha) = d(\delta) = \aleph_0$, there is a bijection $f : D(\delta) \rightarrow D(\alpha)$. We take the $\gamma_3 \in \text{Sym}(X)$ defined by $i\gamma_3 = f(i)$ for $i \in D(\delta)$ and $i\gamma_3 = i\delta^{-1}\eta^{-1}\alpha$ for $i \in \text{im}\delta$. Then for $i \in X$, $i\eta\delta\gamma_3 = (i\eta\delta\delta^{-1}\eta^{-1})\alpha = i\alpha$ since $\text{im}\eta \# \ker \delta$. This shows $\alpha = \eta\delta\gamma_3$ and consequently, $B \subset S$. Altogether, $\text{Inf}(X) = A \cup B \subseteq S$ and thus $S = T(X)$. \square

Lemma 7. *Let $\text{Inj}(X) \subset S \leq T(X)$ with $H \cap S \neq \emptyset$ for $H \in \{\text{Sur}(X), C_p(X), \text{IF}(X)\}$. Then*

$$S = T(X).$$

Proof. We show that then $\text{Sur}(X) \subset S$. If we have $\text{Sur}(X) \subset S$ then from $\text{Inj}(X) \cap S \neq \emptyset$ and $\text{FI}(X) \cap S \neq \emptyset$ (because of $\text{Inj}(X) \subset S$) it follow $S = T(X)$ by Lemma 6.

Let $\alpha \in \text{Sur}(X)$. Moreover, there is a $\beta \in C_p(X) \cap S$. Then there is a bijection

$$f : X/\ker \alpha \rightarrow \{x\beta^{-1} \mid x \in K(\beta)\}.$$

For each $\bar{x} \in X/\ker \alpha$, there is an injective mapping

$$f_{\bar{x}} : \bar{x} \rightarrow f(\bar{x}).$$

We take the $\gamma \in \text{Inj}(X)$ with $i\gamma = f_{\bar{x}}(i)$ where $i \in \bar{x}$ for $\bar{x} \in X/\ker \alpha$. There are $\delta \in \text{Sur}(X) \cap S$ and $\eta \in \text{IF}(X) \cap S$. If $c(\delta) < \aleph_0$ then from $c(\delta) > 0$, it follows $c(\delta^r) > d(\eta)$ for some $r \in \mathbb{N}$, where $\delta^r \in \text{Sur}(X) \cap S$. Hence, we can assume that $c(\delta) > d(\eta)$ and there is a set $A \subseteq X$ with $A \# \ker \delta$ and a bijection

$$h_1 : \text{im}\eta \rightarrow A$$

and an injective mapping

$$h_2 : D(\eta) \rightarrow X \setminus A.$$

We take the $\gamma_1 \in \text{Inj}(X)$ with $i\gamma_1 = h_1(i)$ for $i \in \text{im}\eta$ and $i\gamma_1 = h_2(i)$ for $i \in D(\eta)$. Clearly, $\eta\gamma_1\delta \in \text{Sur}(X) \cap S$ with $c(\eta\gamma_1\delta) = \aleph_0$. So, we can assume that $c(\delta) = \aleph_0$. For $i, j \in X$, $i\alpha = j\alpha$ if and only if there is an $\bar{x} \in X/\ker \alpha$ with $i, j \in \bar{x}$, i.e. $f_{\bar{x}}(i)\beta = f_{\bar{x}}(j)\beta$. But $f_{\bar{x}}(i)\beta = f_{\bar{x}}(j)\beta$ is equivalent to $i\gamma\beta = j\gamma\beta$, consequently, we have $i\gamma\beta = j\gamma\beta$ if and only if $i\alpha = j\alpha$. Further, let $B \subseteq X$ with $B \# \ker \delta$ and

$$\varphi : D(\gamma\beta) \rightarrow X \setminus B$$

be a injective mapping. Then the transformation γ_2 on X with $i\gamma\beta\gamma_2$ is the unique element in $i\alpha\delta^{-1} \cap B$ for $i \in X$ and $i\gamma_2 = \varphi(i)$ for $i \in D(\gamma\beta)$ belongs to $\text{Inj}(X)$. So, we have $i\gamma\beta\gamma_2\delta = i\alpha\delta^{-1}\delta = i\alpha$ for $i \in X$. This shows that $\gamma\beta\gamma_2\delta = \alpha$, and consequently, $\text{Sur}(X) \subset S$. \square

Now we are able to characterize the maximal subsemigroups of $T(X)$ containing $Inv(X)$ and $Sur(X)$, respectively.

Theorem 8. *Let $Sur(X) \subset S \leq T(X)$. Then S is maximal iff $S = T(X) \setminus Inv(X)$ or $S = T(X) \setminus FI(X)$.*

Proof. By Proposition 4, both $T(X) \setminus Inv(X)$ and $T(X) \setminus FI(X)$ are maximal subsemigroups of $T(X)$. Suppose that S is a maximal subsemigroup of $T(X)$. Then $Inv(X) \cap S = \emptyset$ or $FI(X) \cap S = \emptyset$ by Lemma 6, i.e. $S \subseteq T(X) \setminus Inv(X)$ or $S \subseteq T(X) \setminus FI(X)$ and thus $S = T(X) \setminus Inv(X)$ or $S = T(X) \setminus FI(X)$ because of the maximality of S . \square

Theorem 9. *Let $Inv(X) \subset S \leq T(X)$. Then S is maximal iff $S = T(X) \setminus H$ for some $H \in \{Sur(X), C_p(X), IF(X)\}$.*

Proof. By Proposition 4, $T(X) \setminus H$ ($H \in \{Sur(X), C_p(X), IF(X)\}$) are maximal subsemigroups of $T(X)$. If S is a maximal subsemigroup of $T(X)$ then $H \cap S = \emptyset$ for some $H \in \{Sur(X), C_p(X), IF(X)\}$ by Lemma 7, i.e. $S \subseteq T(X) \setminus H$ for some $H \in \{Sur(X), C_p(X), IF(X)\}$. The maximality of S provides the assertion. \square

Finally, we want to determine the maximal subsemigroups of $T(X)$ containing H for $H \in \{C_p(X), IF(X), FI(X)\}$ using Theorem 3. First we state that $FI(X)$ as well as $IF(X)$ are subsemigroups of $T(X)$.

Lemma 10. *$FI(X)$ is a subsemigroup of $T(X)$.*

Proof. Let $\alpha, \beta \in FI(X)$. Then we have $c(\alpha\beta) \leq c(\alpha) + c(\beta) < \aleph_0 + \aleph_0 = \aleph_0$ and $\aleph_0 = d(\beta) \leq d(\alpha\beta)$. This shows that $\alpha\beta \in FI(X)$. \square

Lemma 11. *$IF(X)$ is a subsemigroup of $T(X)$.*

Proof. Let $\alpha, \beta \in IF(X)$. Then we have $d(\alpha\beta) \leq d(\alpha) + d(\beta) < \aleph_0 + \aleph_0 = \aleph_0$ and $\aleph_0 = c(\beta) \leq c(\alpha\beta)$. This shows that $\alpha\beta \in IF(X)$. \square

Let us consider the set $C_p(X) \cap Sur(X)$. Then we have:

Lemma 12. *We have $\langle FI(X), \alpha \rangle = T(X)$ for all $\alpha \in C_p(X) \cap Sur(X)$.*

Proof. Let $\alpha \in C_p(X) \cap Sur(X)$, $\beta \in Inv(X)$, and $A \subseteq X$ be a transversal of α . We put $\gamma \in T(X)$ setting $x\gamma$ is the unique element in $x\beta\alpha^{-1} \cap A$ for all $x \in X$. It is easy to verify that $im\gamma \subseteq A$ and $d(\gamma) = |X \setminus im\gamma| \geq |X \setminus A| = \aleph_0$. Let $x, y \in X$ with $x\gamma = y\gamma$. This implies $(x\beta\alpha^{-1} \cap A)\alpha = (y\beta\alpha^{-1} \cap A)\alpha$, $x\beta = y\beta$, and $x = y$ since $\beta \in Inv(X)$. Thus $\gamma \in Inv(X)$ and $c(\gamma) = 0 < \aleph_0$. Consequently, $\gamma \in FI(X)$. Because of $x\gamma\alpha = (x\beta\alpha^{-1} \cap A)\alpha = x\beta$ for all $x \in X$, we have $\beta = \gamma\alpha \in \langle FI(X), \alpha \rangle$. This shows that $Inv(X) \subseteq \langle FI(X), \alpha \rangle$. Moreover, $\langle FI(X), \alpha \rangle \cap H \neq \emptyset$ for $H \in \{Sur(X), C_p(X), IF(X)\}$. By Lemma 7, we have $\langle FI(X), \alpha \rangle = T(X)$. \square

Lemma 10, Lemma 12 and Theorem 3 imply:

Proposition 13. *Let $S \leq T(X)$ with $FI(X) \subseteq S$. Then the following statements are equivalent:*

(i) *S is maximal.*

(ii) *$S = T(X) \setminus H$ for some $H \in \mathcal{H}(C_p(X) \cap Sur(X), FI(X))$.*

Now, we consider the set $FI(X) \cup Inj(X)$. Here, we get:

Lemma 14. *We have $\langle IF(X), \alpha \rangle = T(X)$ for all $\alpha \in FI(X) \cap Inj(X)$.*

Proof. Let $\alpha \in FI(X) \cap Inj(X)$ and $\beta \in Inj(X)$. We put $\gamma \in T(X)$ setting

$$\begin{aligned} x\alpha\gamma &:= x\beta \text{ for } x \in X \\ x\gamma &:= f(x) \text{ for } x \in D(\alpha) \end{aligned}$$

where

$$f : D(\alpha) \rightarrow D(\beta) \cup x_0\alpha$$

is a surjective transformation such that $|D(\alpha) \setminus \Sigma| = \aleph_0$ for some transversal Σ of f and any fixed $x_0 \in X$. Such a mapping exists because of $d(\alpha) = \aleph_0$. Since $c(f) = \aleph_0$, we have $c(\gamma) = \aleph_0$. Moreover, $im\gamma = \{x\gamma \mid x \in X\} = \{x\gamma \mid x \in im\alpha\} \cup \{x\gamma \mid x \in X \setminus im\alpha\} = im\beta \cup (X \setminus im\beta) \cup \{x_0\alpha\} = X$. Hence $d(\gamma) = 0$. This shows that $\gamma \in IF(X)$. By definition, we have $\beta = \alpha\gamma \in \langle IF(X), \alpha \rangle$. This shows that $Inj(X) \subseteq \langle IF(X), \alpha \rangle$. Moreover, $\langle IF(X), \alpha \rangle \cap H \neq \emptyset$ for $H \in \{Sur(X), C_p(X), IF(X)\}$. By Lemma 7, we have $\langle IF(X), \alpha \rangle = T(X)$. \square

Lemma 11, Lemma 14 and Theorem 3 imply:

Proposition 15. *Let $S \leq T(X)$ with $IF(X) \subseteq S$. Then the following statements are equivalent:*

(i) *S is maximal.*

(ii) *$S = T(X) \setminus H$ for some $H \in \mathcal{H}(Inj(X) \cap FI(X), IF(X))$.*

Lemma 16. $\langle C_p(X) \rangle \cap (Inj(X) \cap FI(X)) = \emptyset$.

Proof. Let $\alpha, \beta \in T(X)$ with $c(\alpha) = c(\beta) = \aleph_0$. Then $\aleph_0 = c(\alpha) \leq c(\alpha\beta)$, i.e. $c(\alpha\beta) = \aleph_0$. Since $c(\alpha) = \aleph_0$ for all $\alpha \in C_p(X)$, this shows that $\langle C_p(X) \rangle \cap FI(X) = \emptyset$. \square

Since we can decompose a countable set into countable many countable sets, it is routine that each transformation α with $\exists \bar{x} \in X/ker\alpha$ with $|\bar{x}| = \aleph_0$ can be written as product $\beta\gamma$ of appropriate transformations $\beta, \gamma \in C_p(X)$. Moreover, it is clear that $\{\alpha \in T(X) \mid \exists \bar{x} \in X/ker\alpha \text{ with } |\bar{x}| = \aleph_0\}$ is subsemigroup of $T(X)$. Hence $\langle C_p(X) \rangle = \{\alpha \in T(X) \mid \exists \bar{x} \in X/ker\alpha \text{ with } |\bar{x}| = \aleph_0\}$.

Lemma 17. *We have $\langle C_p(X), \alpha \rangle = T(X)$ for all $\alpha \in FI(X) \cap Inj(X)$.*

Proof. We show that $Inj(X) \subset \langle C_p(X), \alpha \rangle$. If we have it then from $\langle C_p(X), \alpha \rangle \cap H \neq \emptyset$ for $H \in \{Sur(X), C_p(X), IF(X)\}$ it follows $\langle C_p(X), \alpha \rangle = T(X)$ by Lemma 7. For this let $\beta \in Inj(X)$. Let $\alpha \in S \cap (FI(X) \cap Inj(X))$. Further let $\{I_k \mid k \in X\}$ be a decomposition of $D(\alpha)$ in infinitely many infinite subsets. Then we take $\gamma \in C_p(X) \subset \langle C_p(X), \alpha \rangle$ with $X^+ / \ker \gamma = \{I_k \cup \{k\alpha\} \mid k \in X\}$ and $i\gamma = k\beta$ for $i \in I_k \cup \{k\alpha\}$ and $k \in X$. This provides $k\alpha\gamma = k\beta$ for $k \in X$. This shows $\beta = \alpha\gamma \in S$. Consequently, $Inj(X) \subset \langle C_p(X), \alpha \rangle$. \square

Proposition 18. *Let $S \leq T(X)$ with $C_p(X) \subseteq S$. Then the following statements are equivalent:*

- (i) *S is maximal.*
- (ii) *$S = T(X) \setminus H$ for some $H \in \mathcal{H}(FI(X) \cap Inj(X), \langle C_p(X) \rangle)$.*

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